Complex networks emerging from fluctuating random graphs: Analytic formula for the hidden variable distribution

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In analogy to superstatistics, which connects Boltzmann-Gibbs statistical mechanics to its generalizations through temperature fluctuations, complex networks are constructed from fluctuating Erdös-Rényi random graphs. Using a quantum-mechanical method, the exact analytic formula for the hidden variable distribution is presented which describes the nature of the fluctuations and generates a generic degree distribution through the Poisson transformation. As an example, a static scale-free network is discussed and the corresponding hidden variable distribution is found to decay as a power law and to diverge at the origin.

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In recent years, much effort has been devoted to the investigation of network structures underlying artificial as well as natural complex systems. There, a vertex and an edge represent an element and a relation between elements, respectively. The main focus here is on the topological structure of networks, whereas the physical details of interactions between elements play rather marginal roles. This certainly offers the possibility for an efficient approach to an understanding of collective behaviors in complex systems. Studies along these lines were initiated by the works of Watts and Strogatz [1] on small-world networks and of Barabási and Albert [2] on scale-free networks, see also [3]. These models are fundamentally different from random graphs considered by Erdös and Rényi [4], see also [5], in the following two points: (i) small-world networks have much larger values of the clustering coefficient [1] than random graphs, showing strong correlations between the edges, and (ii) the degree distribution of a scale-free network has the form [2]

$$p_{\mathrm{sf}}(k) \sim \frac{1}{(k+k_0)^{\gamma}} \tag{1}$$

with vertex connectivity, k, and constants, $k_0 \in (0,1)$ and $\gamma > 1$, in contrast to the Poisson distribution for a random graph.

The scale-free degree distribution in Eq. (1) implies that there exist a significant number of vertices that have high values of connectivity (hubs). Such a structure is profoundly relevant to the concept of robustness and vulnerability of the network [6].

In [2] it has been discussed that the so-called preferential attachment rule is sufficient for realizing a growing scale-free network. The rule implies that a vertex created anew tends to be linked to an old vertex with probability proportional to the connectivity of that old vertex. Preferential attachment may be capable of explaining structures such as the Internet [7], the World Wide Web [3], and patterns of citing scientific papers [8], however it does not seem to explain the nature of biological networks such as food webs [9], metabolic networks [10], and protein architectures in cells [11]. This indicates that there should exist a variety of mechanisms, which lead to scale-free statistics in Eq. (1).

There are in fact some methods of generating scale-free networks which do not assume growth and preferential attachment. For example, in a static model given in [12], a weight factor of the specific form is assigned to each vertex. Then, two vertices are randomly selected with the probability proportional to their weights and are connected by an edge if they were not already linked, otherwise they are discarded. Such an algorithm leads to Eq. (1) with $\gamma > 2$ for large k. Another static model proposed in [13] introduces varying vertex fitness. This idea has further been elaborated in [14], see also [15]. Nowadays, varying fitness is referred to as the hidden variable in the literature.

In this paper, we construct static complex networks by making use of fluctuating random graphs. Our discussion, which puts an analytical basis on previous works [12–15], is guided by the spirit of superstatistics [16–20] that establishes a connection between Boltzmann-Gibbs statistical mechanics and its generalizations based on temperature fluctuations. An essential point here is the analogy of Erdös-Rényi random graph theory to Boltzmann-Gibbs statistical mechanics: in the latter, the temperature of a canonical ensemble is fixed, while the probability of connecting each of the two vertices is fixed in the former. Therefore, temperature fluctuations correspond to a varying probability of connection of vertices. Applying a quantum-mechanical method, we derive a general formula for the hidden variable distribution which relates a general complex network to a random graph. As an example, we analyze a scale-free network in detail and present the explicit analytic formula for the associated hidden variable distribution, which is found to decay as a power law. The numerical result presented in [13] can thus be explained analytically.

Let us recall the classical discussion about random graphs of Erdös and Rényi. Consider N vertices. Each pair of vertices is connected by an edge with probability, q. New edges are attached in this way, up to the total number, $N-1$. Then,

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the probability of finding a vertex with degree (connectivity) k is given by the binomial distribution, which in the large-N limit becomes Poissonian,

$$p_{\mathrm{random}}(k) = \frac{\lambda^k}{k!} e^{-\lambda} \quad (k = 0, 1, 2, \ldots), \tag{2}$$

where λ is a fixed constant, $q(N-1)$. This is the analog of the canonical distribution in Boltzmann-Gibbs statistical mechanics, and accordingly λ is seen to correspond most naturally to temperature.

Now, a question is how the degree distribution can change if λ is a stochastic variable, in analogy with temperature fluctuation in superstatistics [16–20]. In this case, the distribution in Eq. (2) is regarded as a conditional probability distribution, $p_{\mathrm{random}}(k|\lambda)$. Thus, for a certain degree distribution, $p(k)$, we have

$$\int_0^\infty d\lambda \Pi(\lambda) \frac{\lambda^k}{k!} e^{-\lambda} = p(k), \tag{3}$$

where $\Pi(\lambda)$ is the associated hidden variable distribution [13–15]. In other words, $p(k)$ is the marginal of the joint probability, $\Pi(\lambda) p_{\mathrm{random}}(k|\lambda)$. Equation (3) implies that $p(k)$ is the Poisson transformation of the hidden variable distribution. In this way, we are able to construct a wide class of complex networks from a fluctuating random graph.

First, let us consider a general form of $p(k)$. This is of obvious importance since real-world networks have diverse structures, but they can still be characterized by their degree distributions [21]. To find the form of $\Pi(\lambda)$, we employ a quantum-mechanical method. In particular, we use the coherent state of the harmonic oscillator with unit mass and frequency. The Planck constant is set equal to unity. The creation and annihilation operators, $\hat{a}^\dagger$ and $\hat{a}$, satisfy the following algebra: $[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$. The ground state is defined by $\hat{a}|0\rangle = 0$, and the k-particle state is given by $|k\rangle = (k!)^{-1/2} (\hat{a}^\dagger)^k |0\rangle$, which is the eigenstate of the number operator, $\hat{n} = \hat{a}^\dagger \hat{a}$, that is, $\hat{n}|k\rangle = k|k\rangle$. $\{|k\rangle\}_{k=0,1,2,\ldots}$ forms an orthonormal complete set, the Fock basis. The coherent state, $|\alpha\rangle$, is defined by $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{k=0}^\infty (\alpha^k / \sqrt{k!}) |k\rangle$, where α is a complex variable. It also forms the (over)complete set, satisfying $\pi^{-1} \int\int d^2\alpha |\alpha\rangle\langle\alpha| = 1$, where $d^2\alpha \equiv d(\mathrm{Re}\,\alpha) d(\mathrm{Im}\,\alpha)$ and the domain of integration is the whole complex α plane. It should be noted that the number distribution in the coherent state is Poissonian: $|\langle k|\alpha\rangle|^2 = [(|\alpha|^2)^k / k!] e^{-|\alpha|^2/2}$. Now, we consider the P representation of a density matrix $\hat{\rho}$ [22,23],

$$\hat{\rho} = \int\int d^2\alpha P(\alpha, \alpha^*) |\alpha\rangle \langle\alpha|. \tag{4}$$

The function $P(\alpha, \alpha^*)$ may be singular and negative in general (however, as we shall see, it turns out to behave well as a probability distribution in the present context). Assume that $\hat{\rho}$ is a certain function of the number operator,

$$\hat{\rho} = p(\hat{n}). \tag{5}$$

Then, it follows from Eq. (4) that

$$\int\int d^2\alpha P(\alpha, \alpha^*) \frac{(|\alpha|^2)^k}{k!} e^{-|\alpha|^2/2} = p(k). \tag{6}$$

In this representation, the degree distribution is the analog of the energy (i.e., particle number) distribution. It may be of interest to develop the second quantization formalism for constructing a network based on such an analogy. Since $\hat{\rho}$ is the function of the number operator, $P(\alpha, \alpha^*)$ is a function only of $|\alpha|^2$. Therefore, with the identification $\lambda \leftrightarrow |\alpha|^2$, we see the parallelism between Eqs. (3) and (6). A crucial point is that it is possible to conversely express $P(\alpha, \alpha^*)$ in Eq. (4) in terms of $\hat{\rho}$ as follows [23]:

$$P(\alpha, \alpha^*) = \frac{1}{\pi^2} e^{|\alpha|^2} \int\int d^2\beta \langle -\beta|\hat{\rho}|\beta\rangle e^{|\beta|^2} e^{\alpha\beta^* - \alpha^*\beta}. \tag{7}$$

From this equation, we can arrive at the general analytic formula for the hidden variable distribution,

$$\Pi(\lambda) = \pi P(\alpha, \alpha^*) = \frac{1}{\pi} e^{|\alpha|^2} \int\int d^2\beta e^{\alpha\beta^* - \alpha^*\beta} \sum_{k=0}^\infty p(k) \frac{(-|\beta|^2)^k}{k!} \tag{8}$$

with $\lambda \equiv |\alpha|^2$.

Finally, let us calculate as an important example the hidden variable distribution for a scale-free network characterized by the degree distribution in Eq. (1). In this case, the normalized density matrix is taken to be

$$\hat{\rho}_{\mathrm{sf}} = \frac{A}{(\hat{n} + k_0)^\gamma}, \tag{9}$$

$$A^{-1} = \zeta(\gamma, k_0), \tag{10}$$

where $\zeta(s, a)$ is Hurwitz's generalized zeta function [24], which is well defined for $s > 1$ (but can be analytically continued to an arbitrary complex s except for the singularity at $s = 1$). Using the Mellin transformation of $\hat{\rho}_{\mathrm{sf}}$, that is, $\hat{\rho}_{\mathrm{sf}} = [\zeta(\gamma, k_0)\Gamma(\gamma)]^{-1} \int_0^\infty dt\, t^{\gamma-1} e^{-(\hat{n}+k_0)t}$ with Euler's gamma

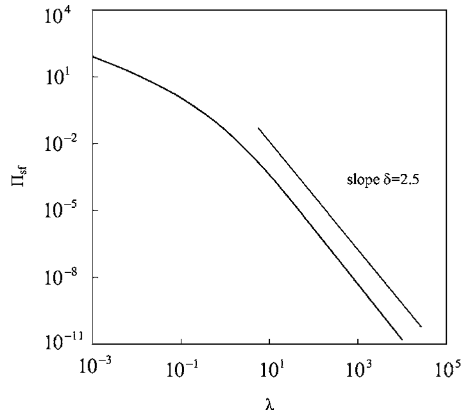

FIG. 1. The log-log plot of $\Pi_{\mathrm{sf}}(\lambda)$ with respect to λ with $\delta(=\gamma) = 2.5$ and $k_0 = 0.5$. All quantities are dimensionless.

function $\Gamma(z)$, we obtain the following analytic formula for the hidden variable distribution:

$$\Pi_{\rm sf}(\lambda) = \frac{1}{\zeta(\gamma,k_0)\Gamma(\gamma)}\int_0^\infty dt t^{\gamma-1} \exp[(1-k_0)t-(e^t-1)\lambda], \tag{11}$$

which may be seen as a main analytic result of the present work for the example of a scale-free network.

The first and second moments of the distribution in Eq. (11) are analytically calculated to be

$$\langle\lambda\rangle = \frac{\zeta(\gamma-1,k_0)}{\zeta(\gamma,k_0)} - k_0, \tag{12}$$

$$\langle\lambda^2\rangle = \frac{\zeta(\gamma-2,k_0)}{\zeta(\gamma,k_0)} - (2k_0+1)\frac{\zeta(\gamma-1,k_0)}{\zeta(\gamma,k_0)} + k_0(k_0+1), \tag{13}$$

respectively. From Eqs. (11)–(13), it is clear that $\Pi_{\rm sf}(\lambda)$ decays as a power law,

$$\Pi_{\rm sf}(\lambda) \sim \frac{1}{\lambda^\delta} \tag{14}$$

with the same value of the exponent as that of the degree distribution

$$\delta = \gamma. \tag{15}$$

However, more significant may be the divergence of $\Pi_{\rm sf}(\lambda)$ in the limit $\lambda \to +0$, which could be observed only through the analytic formula in Eq. (11).

In Fig. 1, we present the plot of $\Pi_{\rm sf}(\lambda)$ with respect to λ, where the asymptotic power law is recognized.

In conclusion, we have constructed in the spirit of superstatistics a generic complex network by introducing fluctuations to random graphs. We have derived the exact analytic formula for the hidden variable distribution which describes the fluctuation and generates a general form of the degree distribution through the Poisson transformation. As an example, we have considered a static scale-free network. We have explicitly calculated the associated hidden variable distribution and have shown that it decays as a power law with the same exponent as that of the degree distribution and diverges at the origin.

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